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COMMENT

A simple derivation of the class operator of the SU(2) group

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Abstract. An elementary derivation of the class operator of the SU(2) group is presented, considerably simplifying the recent calculations of Fan and Ren.

In a recent paper, Fan and Ren (1988) apply their integration technique to obtain, after rather long and complicated calculations, the class operator of the rotation group. In this comment I demonstrate that their result can be obtained within an elementary group-theoretical framework without the use of sophisticated manipulations. Firstly, let us note that the class operator

$$C = \int_{S^2} \mathrm{d}\Omega \exp(\mathrm{i}\psi \boldsymbol{n} \cdot \boldsymbol{J})$$

is SU(2) invariant. Here $\mathrm{d}\Omega$ is the standard measure on the unit sphere S^2, while $\exp(\mathrm{i}\psi \boldsymbol{n} \cdot \boldsymbol{J})$ is an operator representing the rotation through an angle ψ about the axis $\boldsymbol{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi \cos\theta)$. Indeed, for each operator $U(V)$, $V \in \mathrm{SU}(2)$, of a unitary representation of SU(2) under consideration, we have

$$U(V) C U^{\dagger}(V) = \int_{S^2} \mathrm{d}\Omega \exp(\mathrm{i}\psi \boldsymbol{n} \cdot U(V) \boldsymbol{J} U^{\dagger}(V)) = \int_{S^2} \mathrm{d}\Omega \exp[\mathrm{i}\psi (R(V)\boldsymbol{n}) \cdot \boldsymbol{J}] = C$$

because of the transformation properties of the generators J^k under rotation and the invariance of the measure $\mathrm{d}\Omega$; here $R(V)$ is a rotation induced by V. So, for an irreducible representation of SU(2) we have $C = cI$ (I denotes the identity operator) as the consequence of Schur's lemma. Consequently, it is enough to calculate a diagonal element of C, say $\langle m, s|C|m, s\rangle = c$, where $m = -s, -s+1, \ldots, s$ is fixed, while $s = 0, \frac{1}{2}, 1, \ldots$ labels an irreducible representation of SU(2); we have used the standard notation $|m, s\rangle$ for base vectors of the underlying representation space. Explicitly

$$c = \left\langle m, s \left| \int_{S^2} \mathrm{d}\Omega \exp(\mathrm{i}\psi \boldsymbol{n} \cdot \boldsymbol{J}) \right| m, s \right\rangle = \int_{S^2} \mathrm{d}\Omega \langle m, s|\exp(\mathrm{i}\psi \boldsymbol{n} \cdot \boldsymbol{J})|m, s\rangle.$$

Now, the matrix elements of representations of SU(2) are quite well known; they can also be calculated in an elementary way (see e.g. Vilenkin 1968). They take an especially simple form for $m = s$ (Vilenkin 1968, p 116, equation (7)), namely

$$\langle s, s|\exp(\mathrm{i}\psi \boldsymbol{n} \cdot \boldsymbol{J})|s, s\rangle = (\cos\tfrac{1}{2}\psi - \mathrm{i} \sin\tfrac{1}{2}\psi \cos\theta)^{2s}$$

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where we have used the form of $\exp(\mathrm{i}\psi\boldsymbol{n}\cdot\boldsymbol{J})$ in the spinor representation, i.e. $\exp(\mathrm{i}\psi\boldsymbol{n}\cdot\boldsymbol{\sigma}/2)=\cos\frac{1}{2}\psi+\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\frac{1}{2}\psi$, to identify the group parameters. Consequently

$$c=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta(\cos\tfrac{1}{2}\psi-\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta)^{2s}=2\pi\int_{-1}^{1}\mathrm{d}x(\cos\tfrac{1}{2}\psi-\mathrm{i}x\sin\tfrac{1}{2}\psi)^{2s}$$

$$=\frac{4\pi}{(2s+1)\sin\frac{1}{2}\psi}\sum_{k=0}^{[s]}(\cos\tfrac{1}{2}\psi)^{2s-2k}(\sin\tfrac{1}{2}\psi)^{2k+1}(-1)^k\binom{2s+1}{2k+1}.$$

The sum on the right-hand side of the above formula is, however, simply the expression for the expansion of the $\sin[(2s+1)\frac{1}{2}\psi]$ (Gradshteyn and Ryzhik 1980, p 27, equation (1.331.1)); so finally we have

$$c=\frac{4\pi}{(2s+1)}\frac{\sin[(2s+1)\frac{1}{2}\psi]}{\sin\frac{1}{2}\psi}.$$

This is just the formula (30) of Fan and Ren (1988).

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